J.E. AKIN

FINITE ELEMENT ANALYSIS WITH ERROR ESTIMATORS

AN INTRODUCTION TO THE FEM AND
ADAPTIVE ERROR ANALYSIS FOR ENGINEERING STUDENTS

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* Denotes sections or chapters that can be omitted for a first reading or shorter course.

Preface

There are many good texts on the application of finite element analysis techniques. Most do not address the concept and implementation of error estimation. Now that computers are so powerful there is no reason not to carry out a re-analysis until the error levels reach the point that the user is satisfied. Having an error estimation is critical to automating the adaptation of the finite element analysis process. Today several commercial programs include automatic adaptation, based on an error analysis. The user of such programs should have a clear concept of the theory and limitations of such tools. Thus, this text includes the basic finite element theory and its mathematical foundations, the error estimation processes, and the associated computational procedures, as well as several example applications.

This book is primarily intended for advanced undergraduate engineering students and beginning graduate students. The text contains more material than could be covered in a single quarter or semester course. Therefore, a number of chapters or sections that could be omitted in a first course have been marked with an asterisk ($*$). Most of the subject matter deals with linear heat transfer and elementary stress analysis.

The future of finite element analysis will probably heavily involve adaptive analysis methods. One should have a course in Functional Analysis to best understand those techniques. Most undergraduate curriculums do not contain such courses. Therefore, a chapter on mathematical preliminaries is included.

All the Fortran 95 source programs for the general finite element library (called MODEL), and the corresponding application and supporting data file can be downloaded from the World Wide Web (for non-commercial use only). They can be found at the Elsevier site http://www.books.elsevier.com/companions/. The same is true of a large library of small Matlab plotting scripts that display the input and output results shown in the text.

I would like to thank many current and former students at Rice University for their constructive criticisms and comments during the evolution of this book. Special thanks go to Prof. R. L. Taylor, of the University of California at Berkeley for his many detailed and constructive suggestions. Mr. Don Schroder helped with the preparation of a large part of the manuscript. Finally, this book would not have been completed without the support and patience of my wife Kimberly.

Ed Akin
Houston, Texas
2005

Features of the text and accompanying resources

End of chapter exercises
Each chapter ends with a range of exercises that are suitable for homework and assignment work, as well as for private study.

Worked solutions to the exercises are freely available to teachers who adopt or recommend the text to their students. For details on accessing this material please visit http://books.elsevier.com/manuals and follow the registration instructions on screen.

Fortran 95 source programs
Source programs for the general finite element library, and the corresponding application and supporting data file can be freely downloaded from the accompanying website. Go to http://books.elsevier.com/companions and follow the instructions on screen. This material is presented for non-commercial use only.

Matlab plotting scripts library
A library of Matlab plotting scripts that display the input and output results shown in the text are also available for free download from the accompanying website. Go to http://books.elsevier.com/companions and follow the instructions on screen. This material is presented for non-commercial use only.

Notation

The symbols most commonly used throughout the book are defined below. When appearing in the text matrices, tensors, and vectors are identified by boldface type.

Mathematical symbols

$(\hat{.})$	Based on element gradient
$(.^{*})$	Based on nodally continuous gradient
$\{.\}$	Column vector, n by 1
$\lvert . \rvert$	Determinant of a matrix
$\boldsymbol{\Delta}^T$	Divergence operator
$\varnothing$	Empty set
$\boldsymbol{\nabla}$	Gradient operator
$\in$	In
$\bigcap$	Intersection
$[.]^{-1}$	Inverse of a square matrix
$\square$	Non-dimensional parametric space
$\lVert . \rVert$	Norm of a matrix or vector
$].,.[$	Open one-dimensional domain
$\lfloor\ \rfloor^T \lfloor\ \rfloor$	Outer product square matrix, m by m
$,_{(.)}$	Partial differentiation with respect to (.)
$\boldsymbol{\partial}_G, \boldsymbol{\partial}_\Omega$	Partial derivatives in global Cartesian space
$\boldsymbol{\partial}_L, \boldsymbol{\partial}_\square$	Partial derivatives in local parametric space
$\propto$	Proportional to
$[.]$	Rectangular, m by n, or square matrix
$\lfloor . \rfloor$	Row vector, 1 by m
$\subset$	Subset
$[.]^T$	Transpose of a matrix
$\bigcup$	Union

Latin Symbols

A	Area
$\mathbf{a}$	Acceleration vector
a, b, c	Natural coordinates on -1 to $+1$
$(.)^b$	Relating to a boundary domain
$\mathbf{B}$	Differential operator acting on interpolation matrix **H** or **N**
$\mathbf{b}$	Differential operator acting on global interpolation matrix **h**
C^n	Field continuity of degree n
$\mathbf{C}$	System source vector
$\mathbf{C}^b$	Source vector from a boundary segment
$\mathbf{C}^e$	Source vector from an element
$\mathbf{D}$	System degrees of freedom vector
D	Differential operator.
$\mathbf{D}^b$	Boundary segment degrees of freedom vector
$\mathbf{D}^e$	Element degrees of freedom vector
$\mathbf{d}$	Cartesian gradient of **H**
$\mathbf{d}_x$	First row of **d**, etc. for y, z
dof	Degree(s) of freedom
E	Modulus of elasticity of a material
$\mathbf{E}$	Constitutive law (stress-strain) matrix
e	Error
$(.)^e$	Relating to an element domain
$\mathbf{F}$	Resultant force vector
G	Shear modulus of a material
$\mathbf{G}$	Geometry interpolation row matrix (usually $\mathbf{G} = \mathbf{H}$)
$\mathbf{H}^b$	Boundary interpolation row matrix for a scalar
$\mathbf{H}^e$	Element interpolation row matrix for a scalar
h	Characteristic length. Convection coefficient
$\mathbf{h}$	Global interpolation matrix
$I^e, \mathbf{I}^e$	Integral of a scalar or matrix, respectively, on an element
$\mathbf{I}$	Identity matrix
$\mathbf{J}$	Jacobian matrix of a geometric transformation
$\mathbf{K}$	Stiffness matrix
k	Thermal conductivity of a material, or spring stiffness
$\mathbf{L}$	Differential operator
L	Length
L_k	Barycentric coordinates, $\sum L_k = 1$
$\mathbf{M}$	Mass matrix of the system
$\mathbf{m}^e$	Mass matrix, or thermal capacity matrix of an element
m	Mass
$\mathbf{N}$	Interpolation matrix for generalized degrees of freedom (often $\mathbf{N} = \mathbf{H}$)
$\mathbf{n}$	Unit normal vector
n_a	Number of adjacent elements, *NEIGH_L*
n_b	Number of boundary segments, *N_MIXED* + *N_SEG*
n_c	Number of constraint equations, *N_CEQ*

n_d	Number of system degrees of freedom ($n_m \times n_g$), *N_D_FRE*
n_e	Number of elements in the system, *N_ELEMS*
n_f	Maximum number of flux components, *N_G_FLUX*
n_g	Number of generalized dof per node, *N_G_DOF*
n_h	Number of scalar interpolations in **H**, *LT_FREE*
n_i	Number of element equation index terms ($n_n \times n_g$), *LT_FREE*
n_l	Number of elements in a patch, *L_IN_PATCH*
n_m	Maximum node number in the system, *MAX_NP*
n_n	Maximum number of nodes per element, *NOD_PER_EL*
n_o	Number of mixed or Robin BC segments, *N_MIXED*
n_p	Dimension of the parametric space, *N_PARM*
n_q	Number of quadrature points, *N_QP*
n_r	Number of rows in the **B** matrix, *N_R_B*
n_s	Dimension of the physical space, *N_SPACE*
n_t	Number of different element types, *N_L_TYPE*
n_v	Number of vector interpolations in **V**, *LT_FREE*
n_x	Number of element geometry definition nodes, *N_GEOM*
$\mathbf{P}$	Polynomial row matrix. Reaction vector
p	Pressure
$\mathbf{Q}$	Source per unit volume
$\mathbf{Q}^e$	Source per unit volume at element node points
q	Source per unit length
q_n	Heat flux normal to boundary ($\mathbf{q}_n = q_n\,\mathbf{n}$)
$\mathbf{q}$	Heat flux vector at a point
$\mathbf{R}$	Matrix of position vectors, $\mathbf{R} = [\mathbf{x}\,\mathbf{y}\,\mathbf{z}]$
R	Residual error in Ω^e
r, s, t	Unit coordinates on 0 to 1
$\mathbf{S}$	Square matrix of the system
$\mathbf{S}^b$	Square matrix from a boundary segment
$\mathbf{S}^e$	Square matrix from an element
t	Thickness, time
$\mathbf{T}$	Transformation matrix, or boundary traction matrix
U	Strain energy
$\mathbf{u}$	Displacement vector. Velocity vector
u, v, w	Components of displacement vector
V	Volume
$\mathbf{v}$	Velocity vector
W	Mechanical work
x, y, z	Cartesian coordinates
$\mathbf{X}$	Body force vector
$\mathbf{x}$	Vector of x-coordinates
$\mathbf{x}^e$	Vector of x-coordinates of the element nodes
$\mathbf{y}$	Vector of y-coordinates
$\mathbf{z}$	Vector of z-coordinates

Greek symbols

α	Coefficient of thermal expansion
$\boldsymbol{\beta}$	Boolean gather matrix
$\boldsymbol{\beta}^T$	Boolean scatter matrix
$\sum_e \boldsymbol{\beta}^{e^T} \mathbf{C}^e$	Column vector element assembly process
$\sum_e \boldsymbol{\beta}^{e^T} \mathbf{S}^e \boldsymbol{\beta}^e$	Square matrix element assembly process
$\boldsymbol{\Gamma}$	Boundary of a domain, Ω
$\boldsymbol{\Gamma}^b$	Segment of the boundary $\boldsymbol{\Gamma}$
$\boldsymbol{\Gamma}^e$	Boundary of an element domain, Ω^e
γ	Weight per unit volume
$\boldsymbol{\Delta}$	Local derivatives of the interpolation matrix **H** or **N**
$\boldsymbol{\delta}$	Element or boundary segment dof.
$\boldsymbol{\varepsilon}$	Strain or gradient
ζ	Refinement parameter
η	Allowed percentage error
θ	Temperature, or angle
Θ	Effectivity index
λ	Direction cosine wrt x. Lame' constant.
μ	Direction cosine wrt y. Lame' constant.
ν	Poisson's ratio of a material. Direction cosine wrt z.
$\boldsymbol{\Pi}$	Total potential energy, $\boldsymbol{\Pi} = U - W$
π	Mathematical constant 3.14159...
ρ	Mass density of a material
$\boldsymbol{\rho}$	Position vector to a point, $\boldsymbol{\rho} = [x, y, z]$
$\boldsymbol{\sigma}$	Flux or stress
$\boldsymbol{\sigma}^*$	Smoothed flux or stress approximation
$\hat{\boldsymbol{\sigma}}$	Finite element flux or stress approximation
τ	Stabilization parameter
$\boldsymbol{\tau}$	Shear stress
Φ	System degrees of freedom vector
Φ_k	k-th unknown
ϕ	Scalar unknown. Velocity potential
ψ	Stream function
ω	Angular velocity
$\boldsymbol{\Omega}$	Domain
$\boldsymbol{\Omega}^e$	Element domain

Selected program notation (Array sizes follow in parentheses.)

AJ	Jacobian matrix: (N_SPACE, N_SPACE)
AVE	Average quantities at a system node: (N_R_B + 2, MAX_NP)
B	Gradient versus dof matrix: (N_R_B, LT_FREE)
C	Element column matrix: (LT_FREE)
CC	Column matrix of system equations: (N_D_FRE)
COORD	Coordinates of all nodes on an element: (LT_N, N_SPACE)
C_B	Boundary segment column matrix: (LT_FREE)
D	Nodal parameters associated with an element: (LT_FREE)
DD	System list of nodal parameters: (N_D_FRE)
DGH	Global derivatives of scalar functions $\mathbf{H}$: (N_SPACE, LT_N)
DGV	Global derivatives of vector functions $\mathbf{V}$: (N_SPACE, LT_FREE)
DLG	Local derivatives of geometry functions $\mathbf{G}$: (LT_PARM, LT_GEOM)
DLH	Local derivatives of scalar functions $\mathbf{H}$: (LT_PARM, LT_N)
E	Constitutive matrix: (N_R_B, N_R_B)
EL_M	Element mass matrix: (LT_FREE, LT_FREE)
FLUX_LT	Flux at element nodes from a SCP: (SCP_FIT, LT_N
G	Interpolation functions for geometry: (LT_GEOM)
GLOBAL	Global derivatives of scalar interpolation functions $\mathbf{H}$
H	Interpolation functions for an element scalar: (LT_N)
H_INTG	Integral of scalar interpolation functions $\mathbf{H}$: (LT_N)
H_QP	Interpolation for $\mathbf{H}$ at quadrature point: (LT_N, LT_QP)
INDEX	System degree of freedom numbers array: (LT_FREE)
L_B_N	Maximum number of nodes on an element boundary segment
LT	Element type number
LT_FREE	Number of degrees of freedom per element
LT_GEOM	Number of geometric nodes per element
LT_N	Maximum number of nodes for element type
LT_PARM	Dimension of parametric space for element type
LT_QP	Number of quadrature points for element type
LT_SHAP	Current element type shape flag number
L_B_N	Number of nodes on an element boundary segment
L_SHAPE	Shape: 0=Point 1=Line 2=Triangle 3=Quadrilateral 4=Hexahedron 5=Tetrahedron etc.
L_TYPE	Type number array of all elements: (L_S_TOT)
MAT_FLO	Number of real material properties
MAX_NP	Number of system nodes
MISC_FL	Number of miscellaneous floating point (real) system properties
MISC_FX	Number of miscellaneous fixed point (integer) system properties
M_B_N	Number of nodes on a mixed boundagy condition segment
NODES	Node incidences of all elements: (L_S_TOT, NOD_PER_EL)
NOD_PER_EL	Maximum number of nodes per element
N_BS_FIX	Number of boundary segment integer properties
N_BS_FLO	Number of boundary segment real properties
N_CEQ	Number of system constraint equations
N_D_FLUX	Maximum number of flux segment dof = L_B_N * N_G_DOF

N_D_FRE	Total number of system degrees of freedom
N_ELEMS	Number of elements in the system
N_EL_FRE	Maximum number of degrees of freedom per element
N_GEOM	Maximum number of element geometry nodes
N_G_DOF	Number of generalized parameters (dof) per node
N_G_FLUX	Number of flux components per segment node
N_LP_FIX	Number of integer element properties
N_LP_FLO	Number of floating point (real) element properties
N_MAT	Number of materail types
N_MX_FIX	Number of fixed point (integer) mixed segment properties
N_MX_FLO	Number of floating point (real) mixed segment properties
N_NP_FIX	Number of fixed point (integer) nodal properties
N_NP_FLO	Number of floating point (real) nodal properties
N_PARM	Dimension of parametric space
N_PATCH	Number of SCP patches = MAX_NP or N_ELEMS
N_QP	Maximum number of element quadrature points
N_R_B	Number of rows in $\mathbf{B}$ and $\mathbf{E}$ matrices
N_SEG	Number of element boundary segments with given flux
N_SPACE	Dimension of space
PATCH_FIT	Local patch flux values at its nodes: (SCP_N, SCP_FIT)
PT	Quadrature coordinates: (LT_PARM, LT_QP)
S	Element square matrix: (LT_FREE, LT_FREE)
SCP_COUNTS	Number of patches used for each nodal averages: (MAX_NP)
SCP_FIT	Number of terms being fit in a patch, N_R_B usually
SCP_GEOM	Number of patch geometry nodes
SCP_H	Interpolation functions for patch, usually is $\mathbf{H}$ (SCP_N)
SCP_LT	Patch type number
SCP_N	Number of nodes per patch
SCP_PARM	Number of parametric spaces for patch
SCP_QP	Number of quadrature points needed in a SCP patch
SIGMA_HAT	Flux components at a point in original element: (SCP_FIT)
SIGMA_SCP	Flux components at a point in smoothed SCP: (SCP_FIT)
SS	Square matrix of system equations: (N_D_FREE, N_D_FREE)
STRAIN	Strain or gradient vector: (N_R_B + 2)
STRAIN_0	Initial strain or gradient vector, if any: (N_R_B)
STRESS	Stress vector at a point: (N_R_B + 2)
S_B	Boundary segment square matrix, if any: (LT_FREE, LT_FREE)
THIS_EL	Current element number
THIS_LT	Current element type number
THIS_STEP	Current time step number
TIME	Current time in dynamic or transient solution
V	Interpolation functions for vectors: (LT_FREE)
WT	Quadrature weights: (LT_QP)
X	Coordinates of all system nodes: (MAX_NP, N_SPACE)
XYZ	Spatial coordinates at a point: (N_SPACE)

Chapter 1

Introduction

1.1 Finite element methods

The goal of this text is to introduce finite element methods from a rather broad perspective. We will consider the basic theory of finite element methods as utilized as an engineering tool. Likewise, example engineering applications will be presented to illustrate practical concepts of heat transfer, stress analysis, and other fields. Today the subject of error analysis for adaptivity of finite element methods has reached the point that it is both economical and reliable and should be considered in an engineering analysis. Finally, we will consider in some detail the typical computational procedures required to apply modern finite element analysis, and the associated error analysis. In this chapter we will begin with an overview of the finite element method. We close it with consideration of modern programming approaches and a discussion of how the software provided differs from the author's previous implementations of finite element computational procedures.

In modern engineering analysis it is rare to find a project that does not require some type of finite element analysis (FEA). The practical advantages of FEA in stress analysis and structural dynamics have made it the accepted tool for the last two decades. It is also heavily employed in thermal analysis, especially for thermal stress analysis.

Clearly, the greatest advantage of FEA is its ability to handle truly arbitrary geometry. Probably its next most important features are the ability to deal with general boundary conditions and to include nonhomogeneous and anisotropic materials. These features alone mean that we can treat systems of arbitrary shape that are made up of numerous different material regions. Each material could have constant properties or the properties could vary with spatial location. To these very desirable features we can add a large amount of freedom in prescribing the loading conditions and in the post-processing of items such as the stresses and strains. For elliptical boundary value problems the FEA procedures offer significant computational and storage efficiencies that further enhance its use. That class of problems include stress analysis, heat conduction, electrical fields, magnetic fields, ideal fluid flow, etc. FEA also gives us an important solution technique for other problem classes such as the nonlinear Navier-Stokes equations for fluid dynamics, and for plasticity in nonlinear solids.